Solving Nonlinear Evolution Equations Using Physics-Informed Neural Network

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Abstract

Nonlinear evolution equations (NLEEs) model complex physical and biological systems. This study focuses on the application of the Physics-informed neural network (PINN), a robust computational tool, for solving various NLEES, including the Kadomtsev–Petviashvili (KP) equation along with its variants, such as the Jimbo-Miwa equation and others. We presented a detailed study on integrating PINN to solve such NLEEs and enhance the solution's accuracy and computational efficiency. Through various examples from NLEEs, we demonstrate the efficiency of PINN to capture the complex dynamics of such problems and the advantages over traditional numerical methods. The results highlight the advantages of the PINN and provide insights into the broader implications of utilizing the machine learning framework for such complex nonlinear problems.

1 Introduction

Nonlinear partial differential equations are an essential class helpful in modeling complex physical behaviors in a system where linear models are less accurate to consider. They describe the nonlinear interactions of the system, leading to unexpected behavior in the system. These equations are models of various phonemes in fluid dynamics, nonlinear optics, plasma physics, etc. The famous

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Korteweg–de Vries (KdV) equation models shallow water wave propagation, and the modified KdV (mKdV) equation addresses nonlinear wave propagation in polarity symmetry systems. The Kadomtsev–Petviashvili (KP) equation, modified KP (mKP), and potential KP (pKP) equations describe wave propagation in weakly dispersive systems, solitary waves in multi-temperature electron plasma, wave interactions in multidimensional systems with potential term respectively. The nonlinear Schrödinger equation pertains to optical solitons in optical fibers, the Burgers equation characterizes shock waves and acoustic transmission, and the sine-Gordon equation covers fluxon propagation in Josephson junctions between superconductors. The soliton interaction of some of these equations, such as KdV and mKdV, is elastic, where shapes and velocities are not changed after the interaction [32]. However, the soliton interaction of the Burgers equation is an inelastic interaction where fission and fusion may occur.

The study of analytical solutions helps us to clarify the physical properties and behavior of nonlinear equations, which are structural models for many physical phenomena [27]. There are several theoretical approaches to solving analytical solutions, such as the Bäcklund transform [31], inverse scattering method [4], tanh-coth method [2], extended tanh- function method[10], Riemann–Hilbert method [1], Lie group analysis [24], Darboux transformation method [33], the Painlevé analysis [3], the generalized symmetry method [8], Hirota bilinear method [12] and other soliton wave type[26]. Numerical approaches, including Adomian decomposition, homotopy analysis, differential transform methods, finite difference schemes, and many others, offer alternative strategies for solving complex nonlinear equations [29, 7].

Apart from these traditional approaches to solving PDEs, the neural network methods are also successfully applied to solve different PDEs [9, 21, 25]. Among the neural network methods, the physics-informed neural network [25] has been widely applied to solve and learn the PDE models [6, 16]. Even though the PINN shows promising applicability to wide range of problems, the vanilla PINN shows some failure while solving some PDEs, such as high-dimensional, non-linear, multi-scale [30], which leads to the introduction of different versions of PINN such as C-PINN[15], XPINN [14], hp-VPINN [18], and many others.

This work considers the Kadomtsev–Petviashvili (KP) equation and its variants to analyze PINN's applicability and failures while solving such problems using PINN. To discuss further, consider KdV equation derived about a century ago by Korteweg and de Vries [19] derived an equation equivalent to

$$u_t + \alpha u u_x + u_{xxx} = 0, \tag{1}$$

where α is a dispersion coefficient that characterizes the nonlinear interactions within the system. The KdV equation's significance extends to its role in various physical contexts, from ion-acoustic waves in plasma to surface wave propagation. Later, variants of KdV equations are introduced to model different phenomena, and significant strides are made in deriving new solutions and exploring the interactions of solitons.

Through this study, we seek to enhance the understanding of the equation and study the applicability of PINN to solve such equations. Our findings aim to advance theoretical knowledge and offer practical implications for various scientific and engineering disciplines. In this work, we explore the PINN to solve KP equations, its variants, and possible failures while solving such problems. PINN produces smooth solutions, but we can observe sharp changes in many cases of KP solutions. We modify the PINN to capture such sharp changes and adaptively choose the training data to train such models. In Section 2, we discuss the methodology adopted for solving nonlinear partial differential equations (PDEs), outlining the key steps and techniques involved in implementing Physics-Informed Neural Networks (PINNs) for this class of problems. The detailed results, including the performance of the PINN in various test cases and comparisons with traditional numerical schemes, are presented and analyzed in Section 3.

2 Methodology

In mathematical physics, the Kadomtsev–Petviashvili (KP) equation, named after Boris B. Kadomtsev and Vladimir I. Petviashvili, describes nonlinear wave motion [17], which is a two-dimensional generalization of the one-dimensional Korteweg–de Vries (KdV) equation. It studies the dispersive wave stability of the KdV equation in weak transversal perturbations. The KP equation is usually

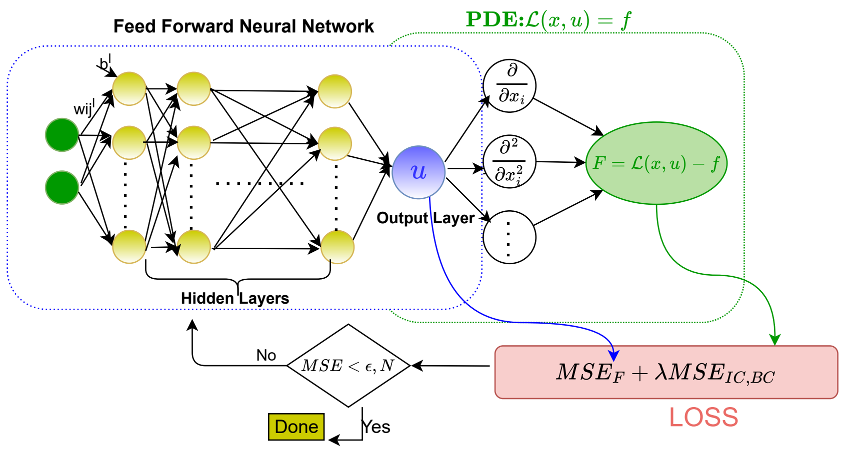


Figure 1: Architecture of the PINN

written as

$$(u_t + 6uu_x + u_{xxx})_x - \sigma u_{yy} = 0, \text{on } \Omega \tag{2}$$

which is classified as the KPI equation when $\sigma = 1$ and the KPII equation when $\sigma = -1$. Later, different extensions of the KP equations were studied in the literature, and solutions were found using different methods [5, 23, 20, 11, 31, 26, 28, 13]. In this study, we aimed to study such equations using PINN. Neural network solutions are promising tools and applications for various problems, making them a promising area of research. In recent years, the advent of machine learning (ML) and deep learning (DL) has opened new avenues for solving PDEs, particularly with the development of Physics-Informed Neural Networks (PINNs). PINN is a promising tool for solving PDEs using information about PDEs and available data. PINNs leverage the power of neural networks (NNs) to approximate solutions to PDEs by directly embedding the governing physical laws into the NN's architecture. This integration allows PINNs to solve PDEs more flexibly and efficiently, particularly in scenarios where traditional methods face limitations. At the heart of a PINN is a neural network that approximates the solution of a PDE. The network takes spatial and temporal coordinates as inputs and outputs the predicted value of the solution at those coordinates. Unlike traditional neural networks, which are trained purely on data, PINNs incorporate the underlying physics by embedding the PDEs into the training process. This is done by adding a term to the loss function that penalizes deviations from the governing equations. The general architecture of the PINN is given in Figure 1.

The loss function in PINNs typically consists of three main components: data-driven loss, physics-informed loss, and boundary and initial conditions loss as given in Equation 3. Data-driven loss minimizes the difference between the NN output and available training data. This is analogous to the loss function in traditional supervised learning. The physics-informed loss term ensures that the network's output satisfies the PDE. It is computed by substituting the network output into the PDE and penalizing residual error. The boundary and initial loss component enforces the boundary and initial conditions of the PDE, ensuring the solution is physically meaningful across the entire domain. The neural network's architecture in PINNs is typically a fully connected feedforward network. The choice of architecture, including the number of layers and neurons per layer, is crucial for the network's ability to capture the solution's complexity. The activation functions used in the network also play a significant role in determining the smoothness and accuracy of the solution. The training process involves optimizing the network parameters using gradient-based optimization methods such as Adam or L-BFGS. A critical advantage of PINNs is automatic differentiation, which allows for the efficient computation of derivatives needed to evaluate the physics-informed loss. For the KP equation, the loss function is defined as for the PINN output u^θ is given by

$$L(\theta, u^\theta) = \|\mathcal{R}_{PDE}[u^\theta]\|_\Omega + \lambda \|\mathcal{R}_{IBC}[u^\theta]\|_{\partial\Omega} \tag{3}$$

$$\mathcal{R}_{PDE}[u^\theta] = (u_t^\theta + 6u^\theta u_x^\theta + u_{xxx}^\theta)_x - \sigma u_{yy}^\theta, \tag{4}$$

where θ is the network parameters, weights, and biases. Then, the PINN algorithm will find a θ^*, the network parameters such that the loss is minimum on these parameters.

One of the significant challenges in solving the KP equation, particularly the KP-I equation, is handling the nonlinear and dispersive terms, which can lead to steep gradients and soliton interactions. PINNs address this by adaptively refining the collocation points—points in the domain where the network is trained to satisfy the PDE—based on the complexity of the solution.

3 Numerical Results and Discussion

One of the significant challenges in solving PDEs using PINNs is dealing with high-dimensional spaces, where traditional methods face the curse of dimensionality. PINNs address this by adaptively refining the sample points in the domain. The effectiveness of the PINN framework is tested on KPI, KPII, and its variant equations. For KPI, known for its challenging dispersive behavior, PINNs are shown to accurately capture soliton interactions and wave dispersion. We will develop a PINN framework tailored to solving the KP equation, analyze its performance compared to the exact solution available, and discuss the advantages and challenges of this approach. Metrics such as accuracy, computational cost, and convergence rates are evaluated.

As a test case, we consider the KPI equations with the exact solution given by

$$u(x,y,t) = \frac{-48(21t^2 - 48xt - 78yt + 12x^2 + 12xy - 24y^2 - 16}{(75t^2 - 48xt + 30yt + 12x^2 + 12xy + 30y^2 + 16)^2}$$

and impose the corresponding initial and boundary conditions. The computational domain is consider as $[-10,10]\times[-10,10]\times[0,1]$. A neural network with 5 hidden layers, each containing 60 neurons, approximates the solution. The training process involves 20000 iterations of the ADAM optimizer followed by 50000 iterations of LBFGS optimization, using 100×100 100×100 data points from the domain and 200 data points for the initial and boundary conditions. After the PINN training, the obtained solution is illustrated in Figure 2, demonstrating the PINN's capability to accurately solve this nonlinear PDE.

We consider the Jimbo-Miwa equation [22] as the second equation in the well-known Kadomtsev-Petviashvili (KP) hierarchy of integrable systems. This equation is particularly significant in nonlinear wave propagation, as it describes certain interesting (3+1)-dimensional waves in physics. Despite its prominent role in modeling wave phenomena, the Jimbo-Miwa equation does not pass many conventional integrability tests, complicating its analytical study. Unlike traditional integrable systems, which are solvable through inverse scattering or the Hirota bilinear method, the Jimbo-Miwa equation presents challenges that prevent it from conforming to these classical frameworks. The equation is given by

$$u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0. \tag{5}$$

These equations model nonlinear wave propagation in multidimensional spaces and Describes physical phenomena such as shallow water waves, plasma waves, and multidimensional soliton interactions. Consider the exact solution given by

$$u(x,y,z,t) = \alpha(z,t) + 2\frac{\tau_x}{\tau}, \tag{6}$$

where,

$$\tau = 1 + e^{ax+f(y,z)+g(t)} + e^{bx+p(y,z)+q(t)} + l(x)e^{(a+b)x+f(y,z)+p(y,z)+g(t)+q(t)},$$

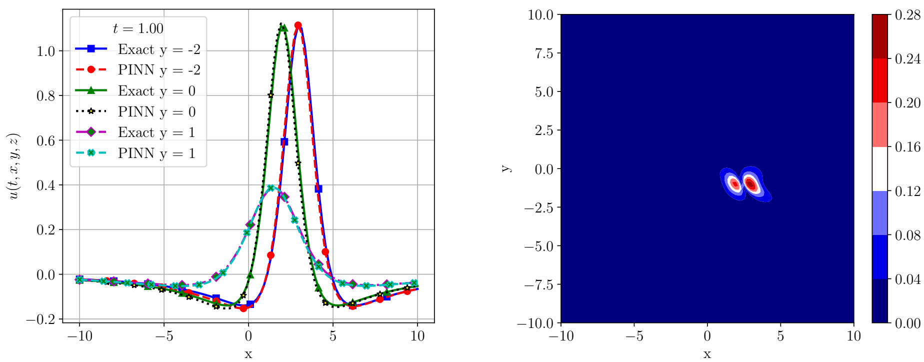
$$f(y,z) = \phi(z + \frac{6by}{4c - b^3 + 3a^2b}), \quad g(t) = \frac{a(a^2b - b^3 + 4c)t}{4b},$$

$$p(y,z) = \psi(z + \frac{3by}{2c + b^3}), \quad q(t) = ct, \quad l(z) = \frac{(a-b)^2}{(a+b)^2},$$

$$a = 2, b = -3, c = -1, \alpha(z,t) = 0,$$

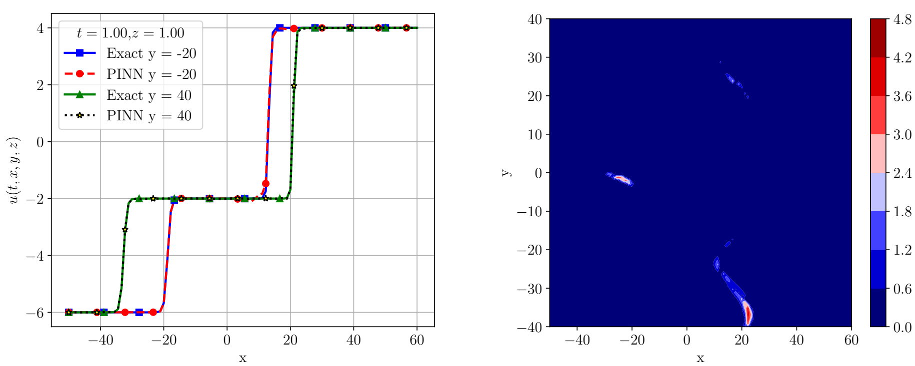
$$\phi(X) = 9\arctan X + 50, \quad \psi(X) = -18\sin(\frac{3X}{4} + 50).$$

The computational domain is considered as $[-50,60]\times[-40,40]\times[0,1]\times[0,1]$. Figure 3 shows this equation's PINN solution and the error plot. The comparison plot highlights the efficiency of the PINN approach, demonstrating its ability to approximate the solution effectively within the given domain. The error plot, representing the difference between the exact and the PINN solutions, underscores areas where the model struggles, especially in regions with sharp gradients or complex nonlinear behavior. This discrepancy emphasizes the challenges of using PINNs for highly nonlinear equations, where the model's efficiency can be limited by the network's architecture and the complexity of the equation itself. The need to mitigate such errors is critical for improving the accuracy and performance of the model.



(a) (b)

Figure 2: Comparison of the exact and PINN solutions for the KP equation and the error distribution. (a) Comparison plot at different y values of the PINN and Exact solution, (b) Error plot.



(a) (b)

Figure 3: Comparison of the exact and PINN solutions for the $(3+1)$ Jimbo-Miwa equation and the error distribution. (a) Comparison plot at different y values of the PINN and Exact solution, (b) Error plot .

4 Conclusion

This study demonstrates that PINNs can effectively solve the Kadomtsev–Petviashvili and its variant equations, capturing complex wave phenomena with high accuracy and stability. Applying Physics-Informed Neural Networks to the KP equation represents a promising new approach to solving such PDEs. PINNs provide a flexible, scalable, and efficient alternative to traditional numerical methods, particularly for high-dimensional problems or those with complex boundary conditions, by embedding the physical laws directly into the neural network framework. While challenges remain, particularly in optimization and computational cost, the potential of PINNs is vast, with ongoing research likely to expand their applicability and effectiveness across a range of scientific and engineering domains. For complex boundary conditions, such as those encountered in real-world applications where the domain may have irregular shapes, the PINN framework can be extended by incorporating domain decomposition techniques or using level-set methods to represent the boundaries.

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References

[1] A B De Monvel and D Shepelsky. A Riemann—Hilbert approach for the Degasperis–Procesi equation. *Nonlinearity*, 26:2081, 2013. doi: {10.1088/0951-7715/26/7/2081}. URL `https://doi.org/10.1088/0951-7715/26/7/2081`.

[2] A M Wazwaz. Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh–coth method. *Appl. Math. Comput.*, 190:633–640, 2007. doi: {10.1016/j.amc.2007.01.056}. URL `https://doi.org/10.1016/j.amc.2007.01.056`.

[3] A M Wazwaz. Painlevé analysis for Boiti–Leon–Manna–Pempinelli equation of higher dimensions with time-dependent coefficients: multiple soliton solutions. *Phys. Lett.*, 384, 2020. doi: {10.1016/j.physleta.2020.126310}. URL `https://doi.org/10.1016/j.physleta.2020.126310`.

[4] Ablowitz, Mark J and Clarkson, Peter A. *Solitons, nonlinear evolution equations and inverse scattering*, volume 149. Cambridge University press, 1991.

[5] Biondini, Gino and Pelinovsky, Dmitry. Kadomtsev-petviashvili equation. *Scholarpedia*, 3(10):6539, 2008.

[6] Cai, Shengze and Wang, Zhicheng and Fuest, Frederik and Jeon, Young Jin and Gray, Callum and Karniadakis, George Em. Flow over an espresso cup: inferring 3-D velocity and pressure fields from tomographic background oriented Schlieren via physics-informed neural networks. *Journal of Fluid Mechanics*, 915:A102, 2021.

[7] Comincioli, Valeriano and Naldi, Giovanni and Scapolla, Terenzio. A wavelet-based method for numerical solution of nonlinear evolution equations. *Applied numerical mathematics*, 33(1-4):291–297, 2000.

[8] D Levi and R I Yamilov. The generalized symmetry method for discrete equations. *J. Phys. A Math. Theor.*, 42, 2009.

[9] Dissanayake, MWMG and Phan-Thien, Nhan. Neural-network-based approximations for solving partial differential equations. *Communications in Numerical Methods in Engineering*, 10(3):195–201, 1994.

[10] Fan, Engui. Extended tanh-function method and its applications to nonlinear equations. *Physics Letters A*, 277(4-5):212–218, 2000.

[11] Grimshaw, Roger and Melville, W K. On the derivation of the modified Kadomtsev-petviashviliequation. *Studies in Applied Mathematics*, 80(3):183–202, 1989.

[12] H D Guo and T C Xia and B B Hu. High-order lumps, high-order breathers and hybrid solutions for an extended (3.1)-dimensional Jimbo–Miwa equation in fluid dynamics. *Nonlinear Dyn.*, 100:601–614, 2020.

[13] Hosseini, K and Hincal, E and Sadri, K and Rabiei, F and Ilie, M and Akgül, A and Osman, MS. The positive multi-complexiton solution to a generalized Kadomtsev–Petviashvili equation. *Partial Differential Equations in Applied Mathematics*, 9:100647, 2024.

[14] Jagtap, Ameya D and Karniadakis, George Em. Extended physics-informed neural networks (XPINNs): A generalized space-time domain decomposition based deep learning framework for nonlinear partial differential equations. *Communications in Computational Physics*, 28(5):2002–2041, 2020.

[15] Jagtap, Ameya D and Kharazmi, Ehsan and Karniadakis, George Em. Conservative physics-informed neural networks on discrete domains for conservation laws: Applications to forward and inverse problems. *Computer Methods in Applied Mechanics and Engineering*, 365:113028, 2020.

[16] Jin, Xiaowei and Cai, Shengze and Li, Hui and Karniadakis, George Em. NSFnets (Navier-Stokes flow nets): Physics-informed neural networks for the incompressible Navier-Stokes equations. *Journal of Computational Physics*, 426:109951, 2021.

[17] Kadomtsev, Boris Borisovich and Petviashvili, Vladimir Iosifovich. On the stability of solitary waves in weakly dispersing Media. In *Doklady Akademii Nauk*, volume 192, pages 753–756. Russian Academy of Sciences, 1970.

[18] Kharazmi, Ehsan and Zhang, Zhongqiang and Karniadakis, George Em. Variational physics-informed neural networks for solving partial differential equations. *arXiv preprint arXiv:1912.00873*, 2019.

[19] Korteweg, Diederik Johannes and De Vries, Gustav. XLI. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 39(240):422–443, 1895.

[20] Kumar, Mukesh and Tiwari, Atul Kumar and Kumar, Raj. Some more solutions of Kadomtsev–Petviashvili equation. *Computers & Mathematics with Applications*, 74(10):2599–2607, 2017.

[21] Lagaris, Isaac E and Likas, Aristidis and Fotiadis, Dimitrios I. Artificial neural networks for solving ordinary and partial differential equations. *IEEE transactions on neural networks*, 9(5):987–1000, 1998.

[22] M Jimbo and T Miwa. Solitons and infinite dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.*, 19:943–1001, 1983. doi: {10.2977/prims/1195182017}. URL `https://doi.org/10.2977/prims/1195182017`.

[23] Ma, Wen-Xiu. Lump solutions to the Kadomtsev–Petviashvili equation. *Physics Letters A*, 379(36): 1975–1978, 2015.

[24] Paquin, G and Winternitz, P. Group theoretical analysis of dispersive long wave equations in two space dimensions. *Physica D: Nonlinear Phenomena*, 46(1):122–138, 1990.

[25] Raissi, Maziar and Perdikaris, Paris and Karniadakis, George E. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational physics*, 378:686–707, 2019.

[26] Rehman, Hamood Ur and Said, Ghada S and Amer, Aamna and Ashraf, Hameed and Tharwat, MM and Abdel-Aty, Mahmoud and Elazab, Nasser S and Osman, MS. Unraveling the (4+ 1)-dimensional Davey-Stewartson-Kadomtsev-Petviashvili equation: Exploring soliton solutions via multiple techniques. *Alexandria Engineering Journal*, 90:17–23, 2024.

[27] S Kara and Ö Ünsal. Implementation of two variable expansion method for extended Jimbo-Miwa equations. *J. Ocean Eng. Sci.*, 2022.

[28] Sağlam, Fatma Nur Kaya and Ahmad, Shabir. Stability analysis and retrieval of new solitary waves of (2+ 1)-and (3+ 1)-dimensional potential Kadomtsev–Petviashvili and B-type Kadomtsev–Petviashvili equations using auxiliary equation technique. *Modern Physics Letters B*, page 2450413, 2024.

[29] Sun, Zhi-Zhong and Zhang, Qifeng and Gao, Guang-hua. *Finite difference methods for nonlinear evolution equations*, volume 8. Walter de Gruyter GmbH & Co KG, 2023.

[30] Wang, Sifan and Teng, Yujun and Perdikaris, Paris. Understanding and mitigating gradient flow pathologies in physics-informed neural networks. *SIAM Journal on Scientific Computing*, 43(5):A3055–A3081, 2021.

[31] Wang, Yi and Lü, Xing. Bäcklund transformation and interaction solutions of a generalized Kadomtsev–Petviashvili equation with variable coefficients. *Chinese Journal of Physics*, 89:37–45, 2024.

[32] Wazwaz, Abdul-Majid. New (3+ 1)-dimensional nonlinear evolution equations with mKdVequation constituting its main part: multiple soliton solutions. *Chaos, Solitons & Fractals*, 76:93–97, 2015.

[33] X Guan and W Liu and Q Zhou and A Biswas. Darboux transformation and analytic solutions for a generalized super-NLS-mKdVequation. *Nonlinear Dyn.*, 98:1491–1500, 2019.